These derivations are quoted directly from Chapter 3, Section 2, beginning on p. 70 of Steven Weinberg's book *Gravitation and Cosmology, Principles and Applications of the General Theory of Relativity*, a John Wiley publication, 1972. Dr. Weinberg states:

"**2. Gravitational Forces**

Consider a particle moving freely under the influence of purely gravitational forces. According to the Principle of Equivalence, there is a freely falling coordinate system ξ^α in which its equation of motion is that of a straight line in space-time, that is,

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{3.2.1}$$

with $d\tau$ the proper time

$$d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{3.2.2}$$

[Compare Eqs. (2.3.1) and (2.1.4).] Now suppose that we use any other coordinate system x^μ, which may be a Cartesian coordinate system at rest in the laboratory, but also may be curvilinear, accelerated, rotating, or what we will. The freely falling coordinates ξ^α are functions of the x^μ, and Eq. (3.2.1) becomes

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \\ &= \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \end{aligned}$$

Multiply this by $\partial x^\lambda / \partial \xi^\alpha$, and use the familiar product rule

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu$$

This gives the equation of motion

$$0 = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{3.2.3}$$

where $\Gamma^\lambda_{\mu\nu}$ is the *affine connection*, defined by

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{3.2.4}$$

The proper time (3.2.2) may also be expressed in an arbitrary coordinate system,

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu \qquad (3.2.5)$$

or

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \qquad (3.2.6)$$

where $g_{\mu\nu}$ is the *metric tensor*, defined by

$$g_{\mu\nu} \equiv \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad (3.2.7)\text{”}$$

This ends the quoted derivation of the geodesic equation taken from Section 2 of Chapter 3 of Dr. Weinberg's book. For a note of clarification of what the $\eta_{\alpha\beta}$ are in Eq. (3.2.2), I further quote from Chapter 2 on Special Relativity, p. 26:

"A Lorentz transformation is a transformation from one system of space-time coordinates x^α to another system x'^α, so that

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha \qquad (2.1.1)$$

where a^α and Λ^α_β are constants, restricted by the conditions

$$\Lambda^\alpha_\gamma\Lambda^\beta_\delta\eta_{\alpha\beta} = \eta_{\gamma\delta} \qquad (2.1.2)$$

with

$$\eta_{\alpha\beta} = \begin{cases} +1 & \alpha=\beta=1,\ 2,\ \text{or } 3 \\ -1 & \alpha=\beta=0 \\ \ \ 0 & \alpha\neq\beta \end{cases} \qquad (2.1.3)$$

In our notation α,β,γ and so on, will always run over the four values 1, 2, 3, 0, with x^1, x^2, x^3 the Cartesian components of the position vector **x** and x^0 the time *t*."

In the first paragraph of Chapter 3, Section 2, Dr. Weinberg states that the coordinate system x^μ "may be a Cartesian coordinate system at rest in the laboratory." Setting the x^μ as such, it is then easy to interpret the metric tensor $g_{\mu\nu}$ given by Eq. (3.2.7) as a metrification of the degree of curvature of the non-Euclidean space-time representing the gravitational field responsible for causing the "falling" of the test particle in the coordinate system x^{μ}, i.e., in the Cartesian laboratory coordinate frame, whose three spatial coordinates *are* Euclidean. Then, the *affine connection* $\Gamma^{\lambda}_{\mu\nu}$ given by Eq. (3.2.4) is understood as the "connection coefficients" between the zero accelerations seen in the "freely falling" frame (see Eq. (3.2.1)) and the non-zero accelerations seen in the laboratory frame, as given by a simple manipulation of Eq. (3.2.3):

$$\frac{d^2 x^{\lambda}}{d\tau^2} = -\Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}$$

If you don't believe that $\Gamma^{\lambda}_{\mu\nu}$ represents a type of "connection" between the freely falling frame and the laboratory frame, you need to convince yourself of the validity of the derivation of Eq. (3.2.4).

It is exactly these laboratory coordinate frame accelerations that need to be numerically integrated in order to numerically map out the generally relativistic motion of a test particle in a gravitational field that pervades the laboratory, and whose metric tensor elements are known functions of the laboratory frame coordinates. One of the great things about the geodesic equation is that it is "coordinate type invariant" meaning that even though we have here interpreted the spatial x^{μ} as the rectangular Cartesian coordinates of the laboratory frame, they can just as well be viewed as their corresponding spherical polar coordinates. Since the Schwarzschild metric tensor, for example, is usually expressed in these polar coordinates, this allows one to immediately derive the laboratory polar coordinate proper time accelerations that need to be numerically integrated to produce full generally relativistic motion in the Cartesian laboratory coordinate frame (see my paper at http://www.mindspring.com/~sb635/pap1.htm).